Friday October 18 Lecture Notes

1 Some Observations About the Transcendence Degree

Let R be a an F-algebra and a domain and suppose trdeg $R < \infty$.

(1) Let us consider K, the filed of fractions of R. Any element of K is algebraic over R because r/s is a root of $s\lambda - r$. So trdeg $R =$ trdeg K.

(2) Suppose A is a nonzero proper ideal of R. Then every nonzero $a \in A$ is transcendental over F because if it were not, then $f_0 = -(\sum_{i=0}^{n} f_i a^{i-1})a \in A$, with $f_i \in F$ and $f_0 \neq 0$. But F is a field, so f_0 is invertible, and so $A = R$, a contradiction.

(3) Suppose A is nonzero proper ideal of R and let R/A be an integral domain. Then every algebraically dependence in R is still a dependence in R/A. So trdeg $R/A \leq$ trdeg R. But we can choose a nonzero $a \in A$, with a transcendental by (2). So it can be extended to a transcendental base, and hence trdeg $R/A <$ trdeg R.

2 Noether's Normalization

Theorem (Noether) Let $R = F[a_1, \ldots, a_n]$ be an affine algebra and suppose trdeg $R = d$. Then there exists $b_1, \ldots, b_n \in R$ such that $R = F[b_1, \ldots, b_n]$ and R is integral over $F[b_1, \ldots, b_n]$.

Proof Permute a_i as necessary so that $a_1, \ldots, a_d$ are algebraically independent. We proceed by induction on n. The case $n = 0$ is trivial (consider $n = d$), so take $n > 0$.

Step 1. We want to construct $R_1 = F[c_1, \ldots, c_{n-1}]$ such that R is integral over R, and $R = R_1[a_n]$. Since $n \geq d$, there exists $f \in F[\lambda_1, \ldots, \lambda_n]$ with $f(a_1, \ldots, a_n) = 0$ where a_n appears nontrivially. Let $c_i = a_i - a_n^r$, with $1 \leq i \leq n-1$ and $r > \deg f$. Then $f(c_1 + a_n^r, \ldots, c_{n-1} + a_n^{r^{n-1}}) = 0$, and the leading term in $f(c_1 + a_n^r, \ldots, c_{n-1} + a_n^{r^{n-1}})$ involves on a_n (by construction). Also, there is not cancellation. So a_n is integral over $R_1 = F[c_1, \ldots, c_{n-1}]$ and hence are the other a_i as $a_i = c_i + a_n^{r^i}$. Thus R is integral over R and $R = R_1[a_n]$.

Step 2. We see that trdeg $R_1 = d$ and so, by induction, $R_1 = F[b_1, \ldots .b_{n-1}]$ is integral over $F[b_1, \ldots, b_d]$. Therefore, by the transitivity of integral extensions, R is integral over $F[b_1, \ldots, b_d]$, and $R = R_1[a_n] = F[b_1, \ldots, b_{n-1}, a_n]$. Now take $a_n = b_n$, and we are done.

3 Maximal Ideals and Prime Ideals

Lemma Let P be a maximal ideal of $F[\lambda_1, \ldots, \lambda_n]$. Then P contains a nonzero irreducible polynomial in $F[\lambda_n]$.

Proof Consider $P \cap F[\lambda_n]$. Then $F[\lambda_n]/P \cap F[\lambda_n]$ can be viewed as subalgebra of $F[\lambda_1, \ldots, \lambda_n]/P$. But P is maximal, so $F[\lambda_1, \ldots, \lambda_n]/P$ is a field and $F[\lambda_n]/P \cap F[\lambda_n]$ is a field, so $P \cap F[\lambda_n]$ is maximal. Finally, maximal ideals in $F[\lambda_n]$ must contain a nonzero irreducible polynomial.

Proposition Let $R = F[\lambda_1, \ldots, \lambda_n]$. Then
(1) Every ideal of the form $L =< \lambda_1 - a_1, \ldots, \lambda_n - a_n >$, for some $a_1, \ldots, a_n \in F$, is maximal.
(2) If F is algebraically closed, then every maximal ideal of R is of that form.

Proof (1) We have $\psi : R \to F$ with $\lambda_i \to a_i$ and $\ker \psi = L$. So $R/L \cong F$, but F is field, which means L is maximal. (2) Let F be algebraically closed and let P be a maximal ideal of F. By the Lemma, P contains an irreducible polynomial in $F[\lambda_n]$. But F is algebraically closed, so the only irreducible polynomials are linear polynomials $\lambda_n - a_n \in P$, say. Let $R_1 = R/ < \lambda_n - a_n >= F[\lambda_1, \ldots, \lambda_{n-1}]$. Let $P_1 = P/ < \lambda_n - a_n >$. Since P_1 is maximal ideal of R_1, so, by induction, $P_1 =< \lambda_1 - a_1, \ldots, \lambda_{n-1} - a_{n-1} >$ and so $P =< \lambda_1 - a_1, \ldots, \lambda_n - a_n >$.

Definition Let R be a commutative ring. An ideal P of R is prime if R/P is an integral domain.

e.g. Every maximal ideal is prime.

e.g. O is prime if and only if R is an integral domain.

Notation: If $I_1, \ldots, I_k$ are ideals of R, then $I_1 \cdots I_k = \{\sum_{\text{finite}} a_{i_1}, \ldots, a_{i_k}, a_{i_j} \in I_j\}$. This is an ideal of R.

Lemma Let R be commutative, and let P be an ideal of R. Then the following are equivalent

(i) P is a prime ideal of R.

(ii) If $a, b \in P$, then $a \in P$ or $b \in P$.
(iii) $R \setminus P$ is closed under multiplication.
(iv) If A and B are ideals of R with $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.
(v) If A and B are ideals of R with $P \subset A$ and $P \subset B$, then $AB \not\subseteq P$.

Proof
(iv) $\iff$ (v) Observe that (v) is the contraposition of (iv).

(i) $\implies$ (ii) Suppose not. If $a, b \in P$, with $a, b \notin P$, then $a + P, b + P \neq 0$ in R/P. But $ab + P = 0$ in R/P, a contradiction.

(ii) $\iff$ (iii) Note that (ii) $\iff [\sim (a \in P \vee b \in P) \implies \sim (ab \in P)] \iff [a \notin P \wedge b \notin P \implies ab \notin P] \iff [a \in R \setminus P \wedge b \in R \setminus P \implies ab \in R \setminus P]$.

(ii) $\implies$ (v) Suppose $A, B \subseteq P$ with $A, B \not\subseteq P$. Take $a, b \in A$ and $a, b \in P$. But $ab \in AB \subseteq P$, a contradiction.

(v) $\implies$ (ii) Take $a, b \in P$ with $a, b \notin P$. Let $A =< a >$ and let $B =< b >$. Then $AB =< ab >$, and so $AB \subseteq P$, but $A, B \not\subseteq P$, a contradiction.

4 Krull Dimension

Definition Let R be commutative. The prime spectrum of R, spec R, is the set of prime ideals of R.

Definition A chain in spec R is an ascending chain $P_0 \subset \cdots \subset P_t$ of length t. A prime P has height t if there is a chain of length t in spec R with P as the largest element, but no such chain of length $t + 1$.

Note: Let I be an ideal of R, then by the 2nd Isomorphism Theorem, there is a set bijection $\{\text{Ideals of } R/I\} \leftrightarrow \{\text{Ideals of } R \text{ containing } I\}$ and $(R/I)/(A/I) \cong R/A$, where A is an ideal of R containing I. So A/I is maximal if and only if A is maximal because $(R/I)/(A/I) \cong R/A$, and A/I is prime if and only if A is prime. This means we have the equivalent set bijection at the level of maximal ideals and of prime ideals. In particular, spec R/I is naturally contained in spec R.

Definition If R is commutative, then Krull dimension of R, $K\text{dim } R$, if it exists, is the maximal height of any prime ideal of R.

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